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Boolean Transfer from Coherent Quantum Logics to Quantum Logics with Continuous Superselection Rules

Hirokazu Nishimura[1]

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Quantum logics with continuous superselection rules are shown to be Boolean-valued coherent quantum logics. Since modern set theory provides a transfer principle from standard mathematics to Boolean-valued mathematics, this makes it possible to transfer automatically well-known results on coherent quantum logics to quantum logics with continuous superselection rules. Many illustrations are given.

1. INTRODUCTION

Without any superselection rules nonrelativistic quantum mechanics is expected to be described by a separable infinite-dimensional Hilbert space over the real numbers **R**, the complex numbers **C**, or the quaternions **Q**. There the propositions correspond to closed subspaces, or equivalently to projections. Observables are represented by self-adjoint operators, and states are in bijective correspondence with von Neumann operators of unit trace. Any automorphism of the propositions is induced by a symmetry, and so forth.

From a lattice-theoretic viewpoint the propositions in the above situation form at least an irreducible complete orthomodular orthocomplemented *AC*-lattice. By a "coherent quantum logic" in the title of this paper we mean such a lattice.

It can be said without exaggeration that quantum mechanics with superselection rules is the rule rather than the exception. This is particularly so in the context of measurements of a quantum system, where we must consider the composition of a quantum system with a classical system. Indeed, as

[1]Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305, Japan.

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Araki (1980) has demonstrated, the notion of a continuous superselection rule lies at the core of Machida and Namiki's (1980) theory of measurement. In such a situation the lattice of propositions is not necessarily even atomistic.

If the superselection rule is discrete, the quantum system decomposes into coherent subsystems, so that the analysis of the system is reducible to the coherent case. What we would like to show in this paper is that the analysis of quantum logics with continuous superselection rules is also reducible to that of coherent quantum logics, though the discussion is much subtler than the discrete case.

The organization of this paper is as follows. In Section 2 we review the rudiments of modern set theory which are requisite for our later discussions. In Section 3 we show that quantum logics with continuous superselection rules are no other than Boolean-valued coherent quantum logics, which enables us to establish a coordinatization theorem for reducible quantum logics in Section 4. There the theory of AW^*-modules invented by Kaplansky (1953) naturally enters, since AW^*-modules are no other than Boolean-valued complex Hilbert spaces, as Ozawa (1984) has demonstrated. In Section 5, just as an operator trace was introduced as a generalization of a trace in the theory of von Neumann algebras, we generalize the notions of an observable and a state, and establish their fundamental theorems, including a generalization of Gleason's theorem to the reducible case, by using Boolean-valued techniques. For other applications of Boolean-valued techniques to mathematics, the reader is referred to Nishimura (1984, 1991), Ozawa (1983, 1984, 1985, 1986), Smith (1984), and Takeuti (1978, 1983), and Takeuti and Zaring (1973).

2. BOOLEAN-VALUED SET THEORY

Let $\mathbf{B}$ be a complete Boolean algebra, which shall be fixed throughout the rest of the paper. We define $V_\alpha^{(\mathbf{B})}$ by transfinite induction on ordinal α as follows:

$$V_0^{(\mathbf{B})} = \varnothing \tag{2.1}$$

$$V_\alpha^{(\mathbf{B})} = \left\{ u \,|\, u\colon \mathscr{D}(u) \to \mathbf{B} \text{ and } \mathscr{D}(u) \subset \bigcup_{\xi<\alpha} V_\xi^{(\mathbf{B})} \right\} \tag{2.2}$$

Then the Boolean-valued universe $V^{(\mathbf{B})}$ of Scott and Solovay is defined as follows:

$$V^{(\mathbf{B})} = \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbf{B})}$$

where On is the class of all ordinal numbers. The class $V^{(\mathbf{B})}$ can be considered to be a Boolean-valued model of set theory by defining $[\![u \in v]\!]$ and $[\![u = v]\!]$ for u, $v \in V^{(\mathbf{B})}$ with simultaneous induction

$$[\![u \in v]\!] = \sup_{y \in \mathscr{D}(v)} (v(y) \wedge [\![u = y]\!]) \tag{2.3}$$

$$[\![u = v]\!] = \inf_{x \in \mathscr{D}(u)} (u(x) \to [\![x \in v]\!]) \wedge \inf_{y \in \mathscr{D}(v)} (v(y) \to [\![y \in u]\!]) \tag{2.4}$$

and by assigning a Boolean value $[\![\alpha]\!]$ to each formula α without free variables inductively as follows:

$$[\![\neg \alpha]\!] = [\![\alpha]\!]^{\perp} \tag{2.5}$$

$$[\![\alpha_1 \vee \alpha_2]\!] = [\![\alpha_1]\!] \vee [\![\alpha_2]\!] \tag{2.6}$$

$$[\![\alpha_1 \wedge \alpha_2]\!] = [\![\alpha_1]\!] \wedge [\![\alpha_2]\!] \tag{2.7}$$

$$[\![\forall x\, \alpha(x)]\!] = \inf_{u \in VB} [\![\alpha(u)]\!] \tag{2.8}$$

$$[\![\exists x\, \alpha(x)]\!] = \sup_{u \in VB} [\![\alpha(u)]\!] \tag{2.9}$$

The following theorem is fundamental to Boolean-valued analysis.

Theorem 2.1. If α is a theorem of ZFC, then so is $[\![\alpha]\!] = 1$.

The class V of all sets can be embedded into $V^{(\mathbf{B})}$ by transfinite induction as follows:

$$\check{y} = \{(\check{x}, 1) \mid x \in y\} \qquad \text{for } y \in V$$

Proposition 2.2. For $x, y \in V$, we have:

(1) $[\![\check{x} \in \check{y}]\!] = \begin{cases} 1 & \text{if } x \in y \\ 0 & \text{otherwise} \end{cases}$

(2) $[\![\check{x} = \check{y}]\!] = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$

A subset $\{e_\lambda\}_{\lambda \in \Lambda}$ of $\mathbf{B}$ is called a *partition of unity* if $\sup_{\lambda \in \Lambda} e_\lambda = 1$ and $e_\lambda \wedge e_{\lambda'} = 0$ for any $\lambda \neq \lambda'$. Given a partition of unity $\{e_\lambda\}_{\lambda \in \Lambda}$ and a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of elements of $V^{(\mathbf{B})}$, one can easily prove the following result.

Theorem 2.3. There exists an element u of $V^{(\mathbf{B})}$ such that $[\![u = u_\alpha]\!] \geq e_\alpha$ for any α. Furthermore, this u is determined uniquely in the sense that $[\![u = v]\!] = 1$ for any $v \in V^{(\mathbf{B})}$ with the above property.

The above u is denoted by $\sum_\alpha u_\alpha e_\alpha$.

We define the *interpretation* $X^{(\mathbf{B})}$ of $X=\{x|\varphi(x)\}$ with respect to $V^{(\mathbf{B})}$ to be $\{u\in V^{(\mathbf{B})}|[\![\varphi(u)]\!]=1\}$, assuming that it is not empty. By way of example, $\mathbf{N}^{(\mathbf{B})}$, $\mathbf{Z}^{(\mathbf{B})}$, $\mathbf{R}^{(\mathbf{B})}$, and $\mathbf{C}^{(\mathbf{B})}$ stand for the totalities of natural numbers, integers, real numbers, and complex numbers in $V^{(\mathbf{B})}$, respectively. We often identify $x\in\mathbf{C}$ with $\check{x}\in\mathbf{C}^{(\mathbf{B})}$.

For technical convenience, if X is a set, then $X^{(\mathbf{B})}$ is usually considered to be a set by choosing a representative from an equivalence class $\{v\in V^{(\mathbf{B})}|[\![u=v]\!]=1\}$. Then we have $X^{(\mathbf{B})}\times\{1\}\in V^{(\mathbf{B})}$ and $[\![X=X^{(\mathbf{B})}\times\{1\}]\!]=1$.

Let $D\subset V^{(\mathbf{B})}$. A function $g: D\to V^{(\mathbf{B})}$ is called *extensional* if $[\![d=d']\!]\leq[\![g(d)=g(d')]\!]$ for any $d, d'\in D$. We say that $u\in V^{(\mathbf{B})}$ is *definite* if $u(d)=1$ for any $d\in\mathscr{D}(u)$. Then we have the following characterization theorem of extensional maps.

Theorem 2.4. Let $u, v\in V^{(\mathbf{B})}$ be definite and $D=\mathscr{D}(u)$. Then there is a bijective correspondence between $h\in V^{(\mathbf{B})}$ satisfying $[\![h: u\to v]\!]=1$ and extensional maps $\varphi: D\to\hat{v}$, where $\hat{v}=\{u|[\![u\in v]\!]=1\}$. The correspondence is given by the relation $[\![h(d)=\varphi(d)]\!]=1$ for any $d\in D$.

A set S with a binary $\mathbf{B}$-valued relation $[\![\cdot=\cdot]\!]$ is called a $\mathbf{B}$-*set* if for any $x, y, z\in S$ we have

$$[\![x=x]\!]=1 \tag{2.10}$$

$$[\![x=y]\!]=[\![y=x]\!] \tag{2.11}$$

$$[\![x=y]\!]\wedge[\![y=z]\!]\leq[\![x=z]\!] \tag{2.12}$$

Given a $\mathbf{B}$-set S, we denote by $\tilde{S}$ the quotient set of $\check{S}$ with respect to the equivalence relation $\{((x, y)\check{}, [\![x=y]\!])|x, y\in S\}$ in $V^{(\mathbf{B})}$. For any $x\in S$, we denote by $\tilde{x}$ the equivalence class of $\check{x}$ with respect to the equivalence relation. Given $u\in V^{(\mathbf{B})}$, we denote by $\hat{u}$ the set $\{v\in V^{(\mathbf{B})}|[\![v\in u]\!]=1\}$, which is considered to be a $\mathbf{B}$-set by choosing representatives.

A $\mathbf{B}$-set is called *separated* if for any $x, y\in S$, $[\![x=y]\!]=1$ implies $x=y$. A $\mathbf{B}$-set S is called *saturated* if for any partition $\{b_i\}_{i\in I}$ of unity of $\mathbf{B}$ and any family $\{x_i\}_{i\in I}\subset S$, there exists $x\in S$ such that $b_i\leq[\![x=x_i]\!]$ for any $i\in I$. A $\mathbf{B}$-set S is called *complete* if it is separated and saturated. For any complete $\mathbf{B}$-set S, we can naturally identify $\hat{\tilde{S}}$ with S. For any nonempty set u in $V^{(\mathbf{B})}$, $\hat{u}$ is a complete $\mathbf{B}$-set, and we can naturally identify $\tilde{\hat{u}}$ with u.

A subset T of a $\mathbf{B}$-set is called a $\mathbf{B}$-*subset* of S if for any $x\in S$, whenever there exist a partition $\{e_\lambda\}_{\lambda\in\Lambda}$ of unity of $\mathbf{B}$ and a family $\{y_\lambda\}_{\lambda\in\Lambda}$ of elements of T such that $[\![x=y_\lambda]\!]\geq e_\lambda$ for any $\lambda\in\Lambda$, we have $x\in T$. A $\mathbf{B}$-subset T of the $\mathbf{B}$-set S is naturally a $\mathbf{B}$-set, and if S is separated, saturated, or complete, then T is so correspondingly. The function $T\mapsto\tilde{T}$ gives a bijective correspondence between the nonempty **B**-subsets of S and the nonempty subsets of $\tilde{S}$ in $V^{(\mathbf{B})}$.

Given two **B**-sets S and T, a function $\varphi: S \to T$ is called a **B**-*function* if for any $x, y \in S$, $[\![x=y]\!] \leq [\![\varphi(x)=\varphi(y)]\!]$. This definition can be extended to n-ary functions for any natural number n. If $\varphi: S \to T$ is a **B**-function, then there exists a unique function $\tilde{\varphi}: \tilde{S} \to \tilde{T}$ in $V^{(\mathbf{B})}$ such that $[\![\tilde{\varphi}(\tilde{x}) = (\varphi(x))^{\sim}]\!] = 1$. For any function $\psi: u \to v$ in $V^{(\mathbf{B})}$, we denote by $\hat{\psi}$ the **B**-function from $\hat{u}$ to $\hat{v}$ such that for any $x \in \hat{u}$, $\hat{\psi}(x)$ is the representative of the class $\{w \in V^{(\mathbf{B})} \mid [\![w = \psi(x)]\!] = 1\}$. If S and T are complete and if we identify S and T with $\hat{\tilde{S}}$ and $\hat{\tilde{T}}$ respectively, then $\varphi \mapsto \tilde{\varphi}$ gives a bijective correspondence between the **B**-functions from S to T and the functions from $\tilde{S}$ to $\tilde{T}$ in $V^{(\mathbf{B})}$ with the inverse $\psi \mapsto \hat{\psi}$.

Given a **B**-set S, the totality S_∞ of families $\{(x_\lambda, e_\lambda)\}_{\lambda \in \Lambda}$ for all partitions $\{e_\lambda\}_{\lambda\in\Lambda}$ of unity of **B** and all families $\{x_\lambda\}_{\lambda\in\Lambda}$ of elements of S becomes a complete **B**-set, provided that we define

$$[\![\{(x_\lambda, e_\lambda)\}_{\lambda\in\Lambda} = \{(y_\delta, f_\delta)\}_{\delta\in\Delta}]\!] = \sup\{[\![x_\lambda = y_\delta]\!] \wedge e_\lambda \wedge f_\delta \mid \lambda \in \Lambda,\ \delta \in \Delta\}$$
$$\text{for} \quad \{(x_\lambda, e_\lambda)\}_{\lambda\in\Lambda},\ \{(y_\delta, f_\delta)\}_{\delta\in\Delta} \in S_\infty$$

and we agree to identify $\{(x_\lambda, e_\lambda)\}_{\lambda\in\Lambda}$ and $\{(y_\delta, f_\delta)\}_{\delta\in\Delta}$ if $[\![\{(x_\lambda, e_\lambda)\}_{\lambda\in\Lambda}, \{(y_\delta, f_\delta)\}_{\delta\in\Delta}]\!] = 1$, i.e., if $[\![x_\lambda, y_\delta]\!] \geq e_\lambda \wedge f_\delta$ for any $\lambda \in \Lambda$ and any $\delta \in \Delta$. If S is separated, then S can naturally be regarded as a **B**-subset of S_∞. $\tilde{S}$ and $\tilde{S}_\infty$ can naturally be identified. S_∞ and $\hat{\tilde{S}}$ can naturally be identified. A **B**-function φ from a **B**-set S to another **B**-set T naturally induces a **B**-function φ_∞ from S_∞ to T_∞, and $\tilde{\varphi}$ and $\tilde{\varphi}_\infty$ can naturally be identified.

The considerations on **B**-sets in this section can be generalized easily to include algebraic structures, so long as algebraic operations are decreed by **B**-functions.

3. QUANTUM LOGICS WITH CONTINUOUS SUPERSELECTION RULES

The techniques of Nishimura (1984) can be used to establish the results of this section, so their proofs are omitted.

An orthocomplemented lattice $\mathscr{L}$ is called a **B**-*orthocomplemented* **B**-*lattice* if it satisfies the following conditions:

1. **B** is an orthosublattice of the center $Z(\mathscr{L})$ of $\mathscr{L}$.
2. For any partition $\{e_\lambda\}_{\lambda\in\Lambda}$ of unity of **B** and any family $\{a_\lambda\}_{\lambda\in\Lambda}$ of elements of $\mathscr{L}$ there exists unique $a \in \mathscr{L}$ such that $a \wedge e_\lambda = a_\lambda \wedge e_\lambda$ for all $\lambda \in \Lambda$.
3. For any $a, b \in \mathscr{L}$ and any $e \in \mathbf{B}$, if $a \wedge e = b \wedge e$, then $a^\perp \wedge e = b^\perp \wedge e$.

In this case $\mathscr{L}$ can naturally be regarded as a **B**-set by defining $[\![a=b]\!]=\sup\{e\in\mathbf{B}\,|\,a\wedge e=b\wedge e\}$ for $a, b\in\mathscr{L}$. By Condition 2 we can see easily that $a\wedge[\![a=b]\!]=b\wedge[\![a=b]\!]$. Regarding $\mathscr{L}$ as a **B**-set, we see that the lattice operations $\wedge$ and $\vee$ and the orthocomplementation $\perp$ are **B**-functions.

Let $\mathscr{L}$ and $\mathscr{L}'$ be **B**-orthocomplemented **B**-lattices. An orthohomomorphism of the orthocomplemented lattice $\mathscr{L}$ into the orthocomplemented lattice $\mathscr{L}'$ is called a **B**-*orthohomomorphism* of $\mathscr{L}$ into $\mathscr{L}'$ if it induces the identity transformation on **B**. Similarly, an orthoisomorphism of the orthocomplemented lattice $\mathscr{L}$ onto the orthocomplemented lattice $\mathscr{L}'$ is called a **B**-*orthoisomorphism* of $\mathscr{L}$ onto $\mathscr{L}'$ if it induces the identity transformation on **B**. If there exists a **B**-orthoisomorphism of $\mathscr{L}$ onto $\mathscr{L}'$, then they are said to be **B**-*orthoisomorphic.*

Proposition 3.1. (1) For any **B**-orthocomplemented **B**-lattice $\mathscr{L}$, $\tilde{\mathscr{L}}$ is an orthocomplemented lattice in $V^{(\mathbf{B})}$, which gives a bijective correspondence between the **B**-orthoisomorphism classes of orthocomplemented **B**-lattices and the orthoisomorphism classes of orthocomplemented lattices in $V^{(\mathbf{B})}$.

(2) For any **B**-orthohomomorphism φ of a **B**-orthocomplemented **B**-lattice $\mathscr{L}$ into a **B**-orthocomplemented **B**-lattice $\mathscr{L}'$, $\tilde{\varphi}$ is an orthohomomorphism of $\tilde{\mathscr{L}}$ into $\tilde{\mathscr{L}}'$, which gives a bijective correspondence between the **B**-orthohomomorphisms of $\mathscr{L}$ into $\mathscr{L}'$ and the orthohomomorphisms of $\tilde{\mathscr{L}}$ into $\tilde{\mathscr{L}}'$ in $V^{(\mathbf{B})}$.

Throughout the rest of this section a **B**-orthocomplemented **B**-lattice $\mathscr{L}$ shall be fixed.

Proposition 3.2. $\mathscr{L}$ is orthomodular iff $\tilde{\mathscr{L}}$ is orthomodular in $V^{(\mathbf{B})}$.

Proposition 3.3. $\mathscr{L}$ is complete (σ-complete) iff $\tilde{\mathscr{L}}$ is complete (σ-complete) in $V^{(\mathbf{B})}$.

An element p of a lattice $\mathscr{L}$ with 0 is called a *quasiatom* if it is an atom or 0. An element p of $\mathscr{L}$ is called a **B**-*quasiatom* if for any $a\in\mathscr{L}$ such that $a\le p$, there exists $e\in\mathbf{B}$ such that $e\wedge a=e\wedge p$ and $e^{\perp}\wedge a=0$. $\mathscr{L}$ is said to be **B**-*atomistic* if every element a of $\mathscr{L}$ is the join of all **B**-quasiatoms p such that $p\le a$.

Lemma 3.4. $p\in\mathscr{L}$ is a **B**-quasiatom of $\mathscr{L}$ iff $\tilde{p}$ is a quasiatom of $\tilde{\mathscr{L}}$ in $V^{(\mathbf{B})}$.

Proposition 3.5. $\mathscr{L}$ is **B**-atomistic iff $\tilde{\mathscr{L}}$ is atomistic in $V^{(\mathbf{B})}$.

$\mathscr{L}$ is said to enjoy the **B**-*covering property* if for any **B**-quasiatom p of $\mathscr{L}$ and any $a\in\mathscr{L}$ we have $(p, a)M$, where $(p, a)M$ means that (p, a) is a modular pair.

Proposition 3.6. $\mathscr{L}$ enjoys the **B**-covering property iff $\tilde{\mathscr{L}}$ enjoys the covering property in $V^{(\mathbf{B})}$.

If the **B**-orthocomplemented **B**-lattice $\mathscr{L}$ is **B**-atomistic and enjoys the **B**-covering property, then $\mathscr{L}$ is called a **B**-*orthocomplemented* **B**-*AC-lattice*.

$\mathscr{L}$ is said to be **B**-irreducible if $Z(\mathscr{L}) = \mathbf{B}$. As in Theorems 4.9 and 4.14 of Nishimura (1984), we have

Proposition 3.7. $\mathscr{L}$ is **B**-irreducible iff $\tilde{\mathscr{L}}$ is irreducible in $V^{(\mathbf{B})}$.

Let n be a natural number. A lattice $\mathscr{L}$ with 0 is said to be of length $\geq n$ if there exist n elements $a_1, \ldots, a_n$ such that $0 < a_1 < \cdots < a_n$. The lattice $\mathscr{L}$ is said to be of **B**-length $\geq n$ if there exist n elements $a_1, \ldots, a_n$ such that $0 < e \wedge a_1 < \cdots < e \wedge a_n$ for any nonzero $e \in \mathbf{B}$.

Proposition 3.8. For a natural number n, $\mathscr{L}$ is of **B**-length $\geq n$ iff $\tilde{\mathscr{L}}$ is of length n in $V^{(\mathbf{B})}$.

An irreducible orthomodular complete orthocomplemented *AC*-lattice is called a *CQL* ("coherent quantum logic"). A **B**-irreducible orthomodular complete **B**-orthocomplemented **B**-*AC*-lattice is called a **B**-QL_{sr} (quantum logic with superselection rule **B**). The preceding results give at once the following theorem.

Theorem 3.9. For any **B**-$\mathrm{QL}_{\mathrm{sr}}$ $\mathscr{L}$, $\tilde{\mathscr{L}}$ is a CQL in $V^{(\mathbf{B})}$, which gives a bijective correspondence between the **B**-orthoisomorphism classes of **B**-$\mathrm{QL}_{\mathrm{sr}}$'s and the orthoisomorphism classes of CQLs in $V^{(\mathbf{B})}$.

4. STANDARD QUANTUM LOGICS WITH CONTINUOUS SUPERSELECTION RULES

The results of this section can be established by using the techniques of Nishimura (1984, 1991), so their proofs are omitted.

In this section a ring always means a ring with unity 1. A *$*$-ring* is a ring $\mathscr{U}$ endowed with a unary operation $*$ such that for any $x, y \in \mathscr{U}$, $x^{**} = x$, $(x+y)^* = x^* + y^*$, and $(xy)^* = y^* x^*$. An element x of a $*$-ring $\mathscr{U}$ is called a *projection* if it is idempotent and self-adjoint, i.e., $x^2 = x$ and $x^* = x$. It is well known that the central projections of a $*$-ring $\mathscr{U}$ form a Boolean algebra, which we denote by $B(\mathscr{U})$. A $*$-ring $\mathscr{U}$ is called a **B**-*$*$-ring* if it satisfies the following conditions:

1. **B** is an orthosublattice of $B(\mathscr{U})$.
2. For any partition $\{e_\lambda\}_{\lambda \in \Lambda}$ of unity of **B** and any family $\{x_\lambda\}_{\lambda \in \Lambda}$ of elements of $\mathscr{U}$ there exists unique $x \in \mathscr{U}$ such that $e_\lambda x = e_\lambda x_\lambda$ for all $\lambda \in \Lambda$.

A **B**-$*$-ring $\mathscr{U}$ can be regarded as a **B**-set by defining $[\![x=y]\!]=\sup\{e\in\mathbf{B}\,|\,ex=ey\}$ for $x, y\in\mathscr{U}$. Two **B**-$*$-rings $\mathscr{U}$ and $\mathscr{U}'$ are called *isomorphic* if there exists a $*$-isomorphism of $\mathscr{U}$ onto $\mathscr{U}'$ inducing the identity transformation on **B**.

Proposition 4.1. If $\mathscr{U}$ is a **B**-$*$-ring, then $\tilde{\mathscr{U}}$ is a $*$-ring in $V^{(\mathbf{B})}$, which induces a bijective correspondence between the isomorphism classes of **B**-$*$-rings and the isomorphism classes of $*$-rings in $V^{(\mathbf{B})}$.

A **B**-$*$-ring $\mathscr{U}$ is called **B**-*division* if any principal left ideal of $\mathscr{U}$ is generated by some element of **B**. If **B** consists only of 0 and 1, the notion of a **B**-division **B**-$*$-ring degenerates into that of a division $*$-ring.

Proposition 4.2. A **B**-$*$-ring $\mathscr{U}$ is **B**-division iff $\tilde{\mathscr{U}}$ is a division $*$-ring in $V^{(\mathbf{B})}$.

Let $\mathscr{U}$ be a **B**-$*$-ring. A module $\mathscr{M}$ over $\mathscr{U}$ is called a **B**-*module* over $\mathscr{U}$ if for any partition $\{e_\lambda\}_{\lambda\in\Lambda}$ of unity of **B** and any family $\{x_\lambda\}_{\lambda\in\Lambda}$ of elements of $\mathscr{M}$ there exists unique $x\in\mathscr{M}$ such that $e_\lambda x=e_\lambda x_\lambda$ for all $\lambda\in\Lambda$. A **B**-module $\mathscr{M}$ over $\mathscr{U}$ can be regarded as a **B**-set by defining $[\![x=y]\!]=\sup\{e\in\mathbf{B}\,|\,ex=ey\}$ for $x, y\in\mathscr{M}$.

Proposition 4.3. Let $\mathscr{U}$ be a **B**-division **B**-$*$-ring. If $\mathscr{M}$ is a **B**-module over $\mathscr{U}$, then $\tilde{\mathscr{M}}$ is a linear space over $\tilde{\mathscr{U}}$, which gives rise to a bijective correspondence between the isomorphism classes of **B**-modules over $\mathscr{U}$ and the isomorphism classes of linear spaces over $\tilde{\mathscr{U}}$ in $V^{(\mathbf{B})}$.

Let $\mathscr{M}$ be a **B**-module over a **B**-$*$-ring $\mathscr{U}$. An element x of $\mathscr{M}$ is called **B**-*nonzero* if, for any $e\in\mathbf{B}$, $ex=0$ implies $e=0$. $\mathscr{M}$ is said to be **B**-*nonzero* if it has a **B**-nonzero element. It is easy to see that an element x of $\mathscr{M}$ is **B**-nonzero iff $\tilde{x}$ is nonzero in $V^{(\mathbf{B})}$, and that $\mathscr{M}$ is **B**-nonzero iff $\tilde{\mathscr{M}}$ is nonzero in $V^{(\mathbf{B})}$. A submodule $\mathscr{N}$ of $\mathscr{M}$ is called a **B**-*submodule* of $\mathscr{M}$ if $\mathscr{N}$ is a **B**-subset of $\mathscr{M}$.

Proposition 4.4. Let $\mathscr{M}$ be a **B**-module over a **B**-division **B**-$*$-ring $\mathscr{U}$. If $\mathscr{N}$ is a **B**-submodule of $\mathscr{M}$, then $\tilde{\mathscr{N}}$ can be regarded naturally as a linear subspace of $\tilde{\mathscr{M}}$ in $V^{(\mathbf{B})}$, which gives rise to a bijective correspondence between the **B**-submodules of $\mathscr{M}$ and the linear subspaces of $\tilde{\mathscr{M}}$ in $V^{(\mathbf{B})}$.

Let $\mathscr{M}$ be a **B**-module over a **B**-division **B**-$*$-ring $\mathscr{U}$. Then a function $\langle\cdot,\cdot\rangle: \mathscr{M}\times\mathscr{M}\to\mathscr{U}$ is called a *Hermitian form* if for any $a_1, a_2\in\mathscr{U}$ and any $x_1, x_2, y\in\mathscr{M}$,

$$\langle a_1x_1+a_2x_2, y\rangle=a_1\langle x_1, y\rangle+a_2\langle x_2, y\rangle \tag{4.1}$$

$$\langle y, x\rangle=\langle x, y\rangle^* \tag{4.2}$$

$$\langle y, y\rangle=0 \qquad \text{implies} \quad y=0 \tag{4.3}$$

Proposition 4.5. Let $\mathscr{M}$ be a **B**-module over a **B**-division **B**-$*$-ring $\mathscr{U}$. If $\langle\cdot,\cdot\rangle$ is a Hermitian form on $\mathscr{M}$, then $\langle\cdot,\cdot\rangle^{\sim}$ is a Hermitian form on the linear space $\tilde{\mathscr{M}}$ over $\tilde{\mathscr{U}}$ in $V^{(\mathbf{B})}$, which gives rise to a bijective correspondence between the Hermitian forms on $\mathscr{M}$ and the Hermitian forms on $\tilde{\mathscr{M}}$ in $V^{(\mathbf{B})}$.

Let $\mathscr{U}$ be a **B**-$*$-ring. Let $\mathscr{M}$ be a **B**-module over $\mathscr{U}$ with a Hermitian form $\langle\cdot,\cdot\rangle$. For any subset X of $\mathscr{M}$ let

$$X^{\perp}=\{y\in\mathscr{M}\,|\,\langle y,x\rangle=0 \text{ for any } x\in X\}$$

A subset X of $\mathscr{M}$ is called *orthoclosed* if $X^{\perp\perp}=X$. It is easy to see that an orthoclosed subset of $\mathscr{M}$ is always a **B**-submodule. We denote by $L_{oc}(\mathscr{M})$ the totality of orthoclosed subsets of $\mathscr{M}$. If **B** consists only of 0 and 1, these considerations degenerate into the well-known case considered, e.g., by Maeda and Maeda (1970, Section 34).

Let $\mathscr{M}$ be a **B**-nonzero **B**-module over a **B**-division **B**-$*$-ring $\mathscr{U}$ with a Hermitian form $\langle\cdot,\cdot\rangle$. Then we have

Theorem 4.6. $L_{oc}(\mathscr{M})$ endowed with the set-theoretic inclusion as its partial ordering and the operation $X\in L_{oc}(\mathscr{M})\mapsto X^{\perp}$ as its orthocomplementation is a **B**-irreducible complete orthocomplemented **B**-AC-lattice, where $e\in\mathbf{B}$ is identified with $\{x\in\mathscr{M}\,|\,(1-e)x=0\}$. We have $(L_{oc}(\mathscr{M}))^{\sim}$ as orthoisomorphic to the orthocomplemented lattice $L_{oc}(\tilde{\mathscr{M}})$ of orthoclosed subspaces of the linear space $\tilde{\mathscr{M}}$ with respect to the Hermitian form $\langle\cdot,\cdot\rangle^{\sim}$ in $V^{(\mathbf{B})}$.

Conversely we have the following result.

Theorem 4.7. Let $\mathscr{L}$ be a **B**-irreducible complete orthocomplemented **B**-AC-lattice of **B**-length ≥ 4. Then there exists a **B**-nonzero **B**-module over a **B**-division **B**-$*$-ring $\mathscr{U}$ with a Hermitian form $\langle\cdot,\cdot\rangle$ such that $\mathscr{L}$ is **B**-orthoisomorphic to $L_{oc}(\mathscr{M})$.

Let $\mathscr{A}$ be a commutative AW^*-algebra whose complete Boolean algebra of projections is **B**. We fix $\mathscr{A}$ throughout the rest of this section. $\mathscr{A}$ can be regarded as a **B**-set by defining $[\![a=b]\!]=\sup\{e\in\mathbf{B}\,|\,ea=eb\}$ for $a,b\in\mathscr{A}$. Ozawa (1984) has shown that $\tilde{\mathscr{A}}$ can be identified with the set of complex numbers in $V^{(\mathbf{B})}$. $\mathscr{A}_{\infty}$ is a **B**-division **B**-$*$-ring by proposition 4.2. A sequence $\{a_i\}_{i\in\mathbf{N}}$ of elements of $\mathscr{A}$ is said to **B**-*converge* to $a\in\mathscr{A}$, written $\lim^{(\mathbf{B})}_{i\to\infty} a_i=a$, if for any $\varepsilon>0$, there exists a partition $\{e_i\}_{i\in\mathbf{N}}$ of unity of **B** such that whenever $j>i$, $\|e_i a_j-e_i a\|<\varepsilon$. It is easy to see that the sequence $\{a_i\}_{i\in\mathbf{N}}$ **B**-converges to a iff its corresponding sequence in $V^{(\mathbf{B})}$ converges to $\tilde{a}$ in $V^{(\mathbf{B})}$. If the derived sequence $\{\sum_{i=1}^{n} a_i\}_{n\in\mathbf{N}}$ **B**-converges to $a\in\mathscr{A}$, then we say that $\sum^{(\mathbf{B})}_{i\in N} a_i$ converges to a.

A module $\mathcal{M}$ over $\mathcal{A}$ is called a *pre-AW^*-module* over $\mathcal{A}$ if it is endowed with a binary function $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ such that the following hold for any $a, b \in \mathcal{A}$ and any $x, y \in \mathcal{M}$:

1. $\langle a_1 x_1 + a_2 x_2, y \rangle = a_1 \langle x_1, y \rangle + a_2 \langle x_2, y \rangle$ for any $a_1, a_2 \in \mathcal{A}$ and any $x_1, x_2, y \in \mathcal{M}$.
2. $\langle y, x \rangle = \langle x, y \rangle^*$ for any $x, y \in \mathcal{M}$.
3. $\langle x, x \rangle \geq 0$ for any $x \in \mathcal{M}$, and the equality holds only if $x = 0$.
4. For any partition $\{e_\lambda\}_{\lambda \in \Lambda}$ of unity of $\mathbf{B}$ and any family $\{x_\lambda\}_{\lambda \in \Lambda}$ of elements of $\mathcal{M}$ with $\sup_{\lambda \in \Lambda} \langle x_\lambda, x_\lambda \rangle < +\infty$, there exists unique $x \in \mathcal{M}$ such that $e_\lambda x = e_\lambda x_\lambda$ for any $\lambda \in \Lambda$.

A pre-AW^*-module $\mathcal{M}$ over $\mathcal{A}$ is called an *AW^*-module* over $\mathcal{A}$ if $\mathcal{M}$ is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$ $(x \in \mathcal{M})$.

Let $\mathcal{M}$ be a pre-AW^*-module over the commutative AW^*-algebra $\mathcal{A}$ with an inner product $\langle \cdot, \cdot \rangle$ taking values in $\mathcal{A}$. One can regard $\mathcal{M}$ as a $\mathbf{B}$-set by defining

$$[\![x = y]\!] = \sup\{e \in \mathbf{B} \mid ex = ey\} \qquad \text{for} \quad x, y \in \mathcal{M}$$

Ozawa (1984) has shown that $\tilde{\mathcal{M}}$ is a complex pre-Hilbert space in $V^{(\mathbf{B})}$, that every complex pre-Hilbert space in $V^{(\mathbf{B})}$ can be obtained in this way up to isomorphism, and that $\tilde{\mathcal{M}}$ is a complex Hilbert space in $V^{(\mathbf{B})}$ iff $\mathcal{M}$ is an AW^*-module. The discussion preceding Theorem 4.6 is applicable to $\mathcal{M}$ with due modifications, and we have the $\mathbf{B}$-orthocomplemented $\mathbf{B}$-lattice $L_{oc}(\mathcal{M})$. It is easy to see that $L_{oc}(\mathcal{M})$ is $\mathbf{B}$-orthoisomorphic to $L_{oc}(\mathcal{M}_\infty)$, where we note that $\mathcal{M}_\infty$ is a $\mathbf{B}$-module over the $\mathbf{B}$-division $\mathbf{B}$-$*$-ring $\mathcal{A}_\infty$ and the inner product on $\mathcal{M}$ naturally induces a Hermitian form on $\mathcal{M}_\infty$. Thus Theorem 4.6 implies that $L_{oc}(\mathcal{M})$ is a $\mathbf{B}$-irreducible complete $\mathbf{B}$-orthocomplemented $\mathbf{B}$-AC-lattice, so long as $\mathcal{M}$ is $\mathbf{B}$-nonzero in the same sense as defined for $\mathbf{B}$-modules over $\mathbf{B}$-$*$-rings. By transferring Amemiya and Araki's (1966) theorem to $V^{(\mathbf{B})}$, we have the following result.

Theorem 4.8. $L_{oc}(\mathcal{M})$ is orthomodular iff $\mathcal{M}$ is an AW^*-module.

Let $\mathcal{M}$ be an AW^*-module over $\mathcal{A}$. A submodule $\mathcal{N}$ of $\mathcal{M}$ is called a $\mathbf{B}$-*submodule* of $\mathcal{M}$ if $\mathcal{N}$ is a $\mathbf{B}$-subset of $\mathcal{M}$. A $\mathbf{B}$-submodule $\mathcal{N}$ of $\mathcal{M}$ is called an *AW^*-submodule* of $\mathcal{M}$ if $\mathcal{N}$ is an AW^*-module with respect to the inherited inner product. It is easy to see that the function $\mathcal{N} \mapsto \tilde{\mathcal{N}}$ gives a bijective correspondence between the $\mathbf{B}$-submodules of $\mathcal{M}$ and the linear subspaces of $\tilde{\mathcal{M}}$ in $V^{(\mathbf{B})}$, under which $\mathcal{N}$ is an AW^*-submodule iff $\tilde{\mathcal{N}}$ is a subspace of $\tilde{\mathcal{M}}$ in $V^{(\mathbf{B})}$. Since the intersection of any family of AW^*-submodules of $\mathcal{M}$ is again an AW^*-submodule, any subset of $\mathcal{M}$ has the minimal AW^*-submodule that contains $\mathcal{M}$. In particular, a submodule $\mathcal{N}$ of $\mathcal{M}$ is said to be $\mathbf{B}$-*dense* if the minimal AW^*-submodule containing $\mathcal{N}$ is $\mathcal{M}$ itself.

In this paper, by an *operator* T on $\mathcal{M}$, we mean a module homomorphism from a **B**-dense submodule $\mathcal{D}(T)$ of $\mathcal{M}$ into $\mathcal{M}$. An operator T is called *symmetric* if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for any $x, y \in \mathcal{D}(T)$. A symmetric operator T is called *self-adjoint* if for any $y \in \mathcal{M}$, so long as there exists $z \in \mathcal{M}$ such that $\langle Tx, y\rangle = \langle x, z\rangle$ for any $x \in \mathcal{D}(T)$, then $y \in \mathcal{D}(T)$.

A surjective operator T is called *unitary* if $\mathcal{D}(T) = \mathcal{M}$ and $\langle Tx, Ty\rangle = \langle x, y\rangle$ for any $x, y \in \mathcal{M}$. In this case T^{-1} is also a unitary operator. A self-adjoint operator T is called a *projection* if $\mathcal{D}(T) = \mathcal{M}$ and $T^2 = T$. Now we have:

Proposition 4.9. (1) For any self-adjoint operator T on $\mathcal{M}$, $\tilde{T}$ is a self-adjoint operator with domain $(\mathcal{D}(T))^{\sim}$ in $V^{(\mathbf{B})}$, which gives a bijective correspondence between the self-adjoint operators on $\mathcal{M}$ and the self-adjoint operators on $\tilde{\mathcal{M}}$ in $V^{(\mathbf{B})}$.

(2) For any unitary operator T on $\mathcal{M}$, $\tilde{T}$ is a unitary operator on $\tilde{\mathcal{M}}$ in $V^{(\mathbf{B})}$, which gives a bijective correspondence between the unitary operators on $\mathcal{M}$ and the unitary operators on $\tilde{\mathcal{M}}$ in $V^{(\mathbf{B})}$.

(3) For any projection operator T on $\mathcal{M}$, $\tilde{T}$ is a projection operator on $\tilde{\mathcal{M}}$ in $V^{(\mathbf{B})}$, which gives a bijective correspondence between the projection operators on $\mathcal{M}$ and the projection operators on $\tilde{\mathcal{M}}$ in $V^{(\mathbf{B})}$.

(4) For any projection operator T on $\mathcal{M}$, the set $\{x \in \mathcal{M} \mid Tx = x\}$ is an AW^*-submodule of $\mathcal{M}$, which gives a bijective correspondence between the projection operators on $\mathcal{M}$ and the AW^*-submodules of $\mathcal{M}$. For any AW^*-submodule $\mathcal{N}$ of $\mathcal{M}$ we write $P^{\mathcal{N}}$ for the projection operator corresponding to $\mathcal{N}$.

Let $\mathcal{M}$ be an $\aleph_0$-homogeneous AW^*-module over $\mathcal{A}$, i.e., there exists a sequence $\{x_i\}_{i \in \mathrm{N}}$ of elements of $\mathcal{M}$ such that $\langle x_i, x_i\rangle = 1$ for any $i \in \mathrm{N}$, $\langle x_i, x_j\rangle = 0$ for $i \neq j$, and the minimal AW^*-submodule containing $\{x_i\}_{i \in \mathrm{N}}$ is $\mathcal{M}$ itself. An operator T on $\mathcal{M}$ with $\mathcal{D}(T) = \mathcal{M}$ is called a *von Neumann operator* if $\langle Tx, x\rangle \geq 0$ for any $x \in \mathcal{M}$ and $\sum_{i \in \mathrm{N}}^{(\mathbf{B})} \langle Tx_i, x_i\rangle$ converges. In this case we write $\mathrm{Tr}\, T$ for $\sum_{i \in \mathrm{N}}^{(\mathbf{B})} \langle Tx_i, x_i\rangle$.

Proposition 4.10. (1) The definition of a von Neumann operator on $\mathcal{M}$ does not depend on the choice of $\{x_i\}_{i \in \mathrm{N}}$.

(2) For any von Neumann operator T on $\mathcal{M}$, $\tilde{T}$ is a von Neumann operator on $\tilde{\mathcal{M}}$, which induces a bijective correspondence between the von Neumann operators of unit trace on $\mathcal{M}$ and the von Neumann operators of unit trace on $\tilde{\mathcal{M}}$ in $V^{(\mathbf{B})}$.

(3) For any von Neumann operator T on $\mathcal{M}$ and a projection operator P on $\mathcal{M}$, PT and TP are von Neumann operators and $\mathrm{Tr}(PT) = \mathrm{Tr}(TP)$.

In the above theorem we warn the reader that not every von Neumann operator on $\tilde{\mathcal{M}}$ is of the form $\tilde{T}$ for a von Neumann operator T on $\mathcal{M}$.

Let $\mathscr{A}_{\mathbf{R}}=\{a\in\mathscr{A}\,|\,a=a^*\}$. It is easy to see that $\tilde{\mathscr{A}}_{\mathbf{R}}$ stands for the real numbers in $V^{(\mathbf{B})}$. Let $\mathscr{A}_{\mathbf{Q}}=\mathscr{A}\times\mathscr{A}$. By defining $(a_1, b_1)(a_2, b_2)=(a_1a_2-b_1b_2^*, a_1b_2+a_2^*b_1)$ and $(a, b)^*=(a^*, -b)$ for any (a_1, b_1), (a_2, b_2), $(a, b)\in\mathscr{A}_{\mathbf{Q}}$, we see that $\mathscr{A}_{\mathbf{Q}}$ becomes a $*$-ring. $\mathscr{A}_{\mathbf{R}}$ and $\mathscr{A}_{\mathbf{Q}}$ are regarded as **B**-sets in the same way as $\mathscr{A}$ is, and $\tilde{\mathscr{A}}_{\mathbf{R}}$ and $\tilde{\mathscr{A}}_{\mathbf{Q}}$ stand for the real and quaternion numbers in $V^{(\mathbf{B})}$, respectively. The notions of an AW^*-module over $\mathscr{A}_{\mathbf{R}}$ and over $\mathscr{A}_{\mathbf{Q}}$ can be defined as expected. The discussions of this section and the following section for AW^*-modules over $\mathscr{A}$ still hold with due modifications for AW^*-modules over $\mathscr{A}_{\mathbf{R}}$ and $\mathscr{A}_{\mathbf{Q}}$, though we deal exclusively with the case over $\mathscr{A}$. A **B**-QL$_{\text{sr}}$ $\mathscr{L}$ is called *standard* if it is **B**-orthoisomorphic to $L_{oc}(\mathscr{M})$ for some $\aleph_0$-homogeneous AW^*-module $\mathscr{M}$ over $\mathscr{A}_{\mathbf{R}}$, $\mathscr{A}$, or $\mathscr{A}_{\mathbf{Q}}$.

5. B-OBSERVABLES AND B-STATES

Let $\mathscr{A}$ be an AW^*-algebra whose complete Boolean algebra of projections is **B** and which shall be fixed throughout this section.

Given two σ-complete **B**-complemented **B**-lattices $\mathscr{L}$ and $\mathscr{L}'$, a σ-orthohomomorphism of $\mathscr{L}$ into $\mathscr{L}'$ is called a **B**-*σ-orthohomomorphism* of $\mathscr{L}$ into $\mathscr{L}'$ if it induces the identity transformation on **B**. As in Proposition 3.1, we have the following result.

Proposition 5.1. For any **B**-σ-orthohomomorphism φ of a σ-complete **B**-orthocomplemented **B**-lattice $\mathscr{L}$ into a σ-complete **B**-orthocomplemented **B**-lattice $\mathscr{L}'$, $\tilde{\varphi}$ is a σ-orthohomomorphism of $\tilde{\mathscr{L}}$ into $\tilde{\mathscr{L}}'$ in $V^{(\mathbf{B})}$, which gives a bijective correspondence between the **B**-σ-orthohomomorphisms of $\mathscr{L}$ into $\mathscr{L}'$ and the σ-orthohomomorphisms of $\tilde{\mathscr{L}}$ into $\tilde{\mathscr{L}}'$ in $V^{(\mathbf{B})}$.

Let $\mathscr{L}$ be a **B**-QL$_{\text{sr}}$. A **B**-*observable* on $\mathscr{L}$ is a **B**-σ-orthohomomorphism of $\mathscr{B}(\mathbf{R})^{(\mathbf{B})}$ into $\mathscr{L}$, where we recall that $\mathscr{B}(\mathbf{R})^{(\mathbf{B})}$ stands for the Borel sets of real numbers in $V^{(\mathbf{B})}$, and by Propositions 3.1 and 3.3 it is a σ-complete **B**-orthocomplemented **B**-lattice. Then proposition 5.1 gives the following result.

Proposition 5.2. For any **B**-observable φ on $\mathscr{L}$, $\tilde{\varphi}$ is an observable on $\tilde{\mathscr{L}}$ in $V^{(\mathbf{B})}$, which gives rise to a bijective correspondence between the **B**-observables on $\mathscr{L}$ and the observables on $\tilde{\mathscr{L}}$ in $V^{(\mathbf{B})}$.

By transferring the spectral theorem for self-adjoint operators to $V^{(\mathbf{B})}$, we have the following.

Theorem 5.3. Let $\mathscr{L}$ be $L_{oc}(\mathscr{M})$ for some **B**-nonzero AW^*-module over $\mathscr{A}$. Then for any **B**-observable φ on $\mathscr{L}$ there exists a unique self-adjoint operator T on $\mathscr{M}$ such that

$$\varphi([r, s]^{(\mathbf{B})}) = \{x \in \mathscr{D}(T) \mid r\langle x, x\rangle \leq \langle Tx, x\rangle \leq s\langle x, x\rangle\} \qquad \text{for any} \quad r, s \in \mathbf{R}$$

with $[r, s]^{(\mathbf{B})} = \{t \in \mathbf{R}^{(\mathbf{B})} \mid r \leq t \leq s \text{ in } V^{(\mathbf{B})}\}$, which gives a bijective correspondence between the **B**-observables on $\mathscr{L}$ and the self-adjoint operators on $\mathscr{M}$.

Let $\mathscr{L}$ be a **B**-QL$_{\text{sr}}$. A **B**-*state* on $\mathscr{L}$ is a function α of $\mathscr{L}$ into $\mathscr{A}_{\mathbf{R}}$ satisfying the following conditions:

1. $0 \leq \alpha(x) \leq 1$ for any $x \in \mathscr{L}$.
2. $\alpha(e \wedge x) = e\alpha(x)$ for any $e \in \mathbf{B}$ and $x \in \mathscr{L}$.
3. $\alpha(1) = 1$.
4. For any orthogonal sequence $\{x_i\}_{i \in N}$ of $\mathscr{L}$, $\alpha(\bigvee_{i \in \mathbf{N}} x_i) = \sum_{i \in \mathbf{N}}^{(\mathbf{B})} \alpha(x_i)$.

It is easy to see that Conditions 2 and 3 imply $\alpha(e) = e$ for any $e \in \mathbf{B}$, since $\alpha(e) = \alpha(e \wedge 1) = e\alpha(1) = e$. It is also easy to see the following result.

Proposition 5.4. For any **B**-state α on a **B**-QL$_{\text{sr}}$ $\mathscr{L}$, $\tilde{\alpha}$ is a state on $\tilde{\mathscr{L}}$ in $V^{(\mathbf{B})}$ which gives a bijective correspondence between the **B**-states on $\mathscr{L}$ and the states on $\tilde{\mathscr{L}}$ in $V^{(\mathbf{B})}$.

Let $\mathscr{L}$ be $L_{oc}(\mathscr{M})$ for some $\aleph_0$-homogeneous AW^*-module over $\mathscr{A}$. It is easy to see that for any von Neumann operator U of unit trace on $\mathscr{M}$, $\alpha_U: \mathscr{N} \in \mathscr{L} \to \mathrm{Tr}(P^{\mathscr{N}} U)$ is a **B**-state. By transferring Gleason's (1957) theorem to $V^{(\mathbf{B})}$, we have the following result.

Theorem 5.5. $U \mapsto \alpha_U$ gives a bijective correspondence between the von Neumann operators of unit trace on $\mathscr{M}$ and the **B**-states on $\mathscr{L}$.

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